The Equations of Bitangential Curves of the General Plane Quintic and Sextic Curves.

THESIS

Presented to the Faculty of the Graduate School

of the

UNIVERSITY OF PENNSYLVANIA

In Partial Fulfillment of the Requirements for the Degree

of Doctor of Philosophy.

1905

By ULYSSES S. HANNA.

ACAD. LUGD. BAT. BIBL.

Philadelphia

THE EQUATIONS OF BITANGENTIAL CURVES OF THE GENERAL PLANE QUINTIC AND SEXTIC CURVES.

By **Ulysses S. Hanna** (Bloomington, Ind.).

Estratto dal tomo XXVIII (2° sem. 1909) dei ***Rendiconti del Circolo Matematico di Palermo.***
Adunanza del 14 febbrajo 1909.

Introduction.

The problems of determining the number of bitangents to a curve of the n-th order and finding the equation of a bitangential curve of the general plane n-ic curve were studied by Plücker [1]), Jacobi [2]) and Clebsch [3]). They found the number of bitangents to be $\frac{1}{2}n(n-2)(n^2-9)$ and the degree of the equation of the bitangential of the general plane n-ic curve to be $(n-2)(n^2-9)$. Hesse [4]) was the first to publish the equation of a bitangential of the general plane quartic curve. Dersch [5]) and Freyberg [6]) have also published equations of bitangentials of the quartic. Maisano [7]) has given us the equation of a bitangential of the general plane quintic curve. All these writers have followed, in general, the method indicated by Jacobi and Clebsch and have used the so called symbolic notation.

---

[1]) *Solution d'une question fondamentale concernant la théorie générale des courbes* [Journal für die reine und angewandte Mathematik, Vol. XII (1834), pp. 105-108]. — *Theorie der algebraischen Curven* (Bonn 1839).

[2]) *Beweis des Satzes dass eine Curve n^{ten} Grades im Allgemeinen $\frac{1}{2}n(n-2)(n^2-9)$ Doppeltangenten hat* [Journal für die reine und angewandte Mathematik, Vol. XL (1850), pp. 237-260].

[3]) *Bemerkung zu* Jacobi's *Beweis für die Anzahl der Doppeltangenten* [Journal für die reine und angewandte Mathematik, Vol. LXIII (1864), pp. 186-188].

[4]) *Über Curven dritter Ordnung und die Kegelschnitte, welche diese Curven in drei verschiedenen Puncten berühren* [Journal für die reine und angewandte Mathematik, Vol. XXXVI (1848), pp. 143-176], p. 156. — *Brief an* Jacobi [Ibid., Vol. XL (1850), p. 260]. — *Über die ganzen homogenen Functionen von der dritten und vierten Ordnung zwischen drei Variabeln* [Ibid., Vol. XLI (1851), pp. 285-292], p. 292.

[5]) *Doppeltangenten einer Curve n^{ter} Ordnung* [Mathematische Annalen, Vol. VII (1874), pp. 497-511].

[6]) *Die Gleichung für die Berührungspunkte der Doppeltangenten der Curve 4. Ordnung* [Mathematische Annalen, Vol. XVII (1880), pp. 329-331].

[7]) *Gleichung der Curve, welche die Berührungspunkte der doppelten Tangenten der allgemeinen Curve des fünften Grades ausschneidet* [Mathematische Annalen, Vol. XXIX (1887), pp. 431-446].

SALMON [8]) and CAYLEY [9]) also have developed methods for finding equations of bitangential curves. They succeeded, however, in applying their theory only in the case of the quartic curve. The results of their endeavors to find equations of bitangential curves of the quintic curve are given by SALMON in «*Higher Plane Curves*». HEAL [10]) succeeded in completing the work for the equation of a bitangential of the quintic after one of the methods indicated by SALMON, except that the coefficients of his quintic are restricted by one condition that must be imposed. In his work he encounters the square of the hessian of the quintic as an extraneous factor and in the removal of this factor he used a formula which he did not prove. The proof will be given in connection with the case of the quintic in this paper.

We shall work with the equation of the general plane n-ic curve in the symbolic form

$$(a_1 x_1 + a_2 x_2 + a_3 x_3)^n = a_x^n = b_x^n = c_x^n = \cdots = 0,$$

and apply the results obtained to the cases $n = 4$ to $n = 8$ inclusive. The method followed will differ very materially from the methods used in the papers cited above and will give us an extraneous factor in each bitangential except that of the quartic. This factor is the $(n-3)(n-4)^{\text{th}}$ power of the hessian of the n-ic. It is easily removed in the cases of the quintic and the sextic and in the case of the septimic I believe it can be removed without much difficulty. In the case of the octic it can be removed but the labor necessary to do it would be very tedious to perform.

§ 1.

General method.

The equation of the general plane n-ic curve written in the form

$$a_x^n = f_1 x_1 + f_2 x_2 + f_3 x_3 = f_x = 0$$

will permit us to use the abbreviations

$$f_1 = a_x^{n-1} a_1, \quad f_2 = a_x^{n-1} a_2, \quad f_3 = a_x^{n-1} a_3,$$

where f_1, f_2 and f_3 differ from the first derivatives of f_x with respect to x_1, x_2 and x_3 by the factor n. The first polar of $f_x = 0$ with respect to the point y has for its equation

$$a_x^{n-1} a_y = f_1 y_1 + f_2 y_2 + f_3 y_3 = f_y = 0.$$

---

[8]) *On the Double Tangents to Plane Curves* [Philosophical Magazine, 4 Series, Vol. XVI (1858), pp. 318-319].

[9]) *On the Double Tangents of a Plane Curve* [Philosophical Transactions of the Royal Society of London, Vol. CXLIX (1859), pp. 193-212].

[10]) *The Bitangential of the Quintic* [Annals of Mathematics, Vol. V (1889-1890), pp. 33-41; Vol. VI (1891-1892), pp. 64-68].

We shall also use the notation

$$f_{11} = a_x^{n-2} a_1^2, \quad f_{12} = a_x^{n-2} a_1 a_2, \quad f_{23} = a_x^{n-2} a_2 a_3, \text{ etc.},$$

for the purpose of finding symbolic expressions for certain covariants that will appear. It will happen in some cases that the numerical coefficients n and $n(n-1)$, encountered in finding the first and second derivatives of f_x, will be retained. This will be indicated in each case so that no confusion will arise.

The point $\lambda x + \mu y$ is any point on the straight line connecting the points x and y. The condition that the point $\lambda x + \mu y$ shall lie upon the n-ic curve, represented by the equation

$$a_z^n = 0$$

is the equation

$$(1) \qquad (a_{\lambda x+\mu y})^n = (\lambda a_x + \mu a_y)^n = 0.$$

The conditions that the point x shall be a point on the n-ic curve and that the point y shall be a point on the tangent to the n-ic curve at the point x are the equations

$$f_x = 0, \quad f_y = 0.$$

Assuming these two conditions and at the same time the condition expressed by equation (1), we have the equation

$$(2) \qquad \begin{cases} A_0\lambda^{n-2} + (n-2)A_1\lambda^{n-3}\mu + \dfrac{(n-2)(n-3)}{2!}A_2\lambda^{n-4}\mu^2 + \cdots \\ \qquad \cdots + (n-2)A_{n-3}\lambda\mu^{n-3} + A_{n-2}\mu^{n-2} = 0, \end{cases}$$

where, for brevity, we have put

$$(3) \quad A_0 = \frac{n(n-1)}{1.2} a_x^{n-2} a_y^2, \quad A_1 = \frac{n(n-1)}{2.3} a_x^{n-3} a_y^3, \ldots, A_r = \frac{n(n-1)}{(r+1)(r+2)} a_x^{n-r-2} a_y^{r+2}.$$

The equation (2), as an equation in λ and μ, determines the positions of the point $\lambda x + \mu y$ where the tangent to the n-ic curve at the point x again cuts the n-ic curve. The discriminant of equation (2), placed equal to zero, is the equation of a bitangential curve of the general plane n-ic curve, containing, however, the coordinates of the arbitrary point y as well as the coordinates of the point of bitangency x. If it is possible to remove from the equation the coordinates of this arbitrary point y on the bitangent, or to express them in terms of the coordinates of the point of bitangency x, we shall be able to find the equation sought.

To do this, the point y is assumed to be the intersection of the tangent and an arbitrary line represented by the equations

$$f_y = 0, \quad u_y = 0.$$

We thus have

$$y_i = (fu)_i,$$

where we have put

$$(fu)_1 = f_2 u_3 - f_3 u_2, \text{ etc.}$$

From this result we get

$$(4) \quad a_y = a_1 y_1 + a_2 y_2 + a_3 y_3 = a_1 (fu)_1 + a_2 (fu)_2 + a_3 (fu)_3 = (afu) = b_x^{n-1} (abu).$$

Substituting this determinant for the a_y in the discriminant of the equation (2), our problem becomes that of the removal of the u instead of the y. It has been shown by JACOBI and CLEBSCH that the removal of the u is not impossible in any case.

§ 2.

A departure from the methods of earlier papers.

Instead of forming the discriminant of equation (2) and then attempting to remove the u therefrom, which is the method used in the papers on the quartic and quintic referred to above, we shall first operate on the equation (2) the substitution

$$(S_1) \qquad \begin{cases} \lambda = \lambda_1 - A_1 \mu_1 \\ \mu = \quad A_0 \mu_1 , \end{cases}$$

the determinant of which is

$$\begin{vmatrix} 1 & -A_1 \\ 0 & A_0 \end{vmatrix} = A_0 ,$$

and then form the discriminant of the equation thus found, after first having removed the u from each of the coefficients of the new equation separately.

Let us suppose that the equation gotten by this substitution is

$$(5) \qquad \begin{cases} B_0 \lambda_1^{n-2} + (n-2) B_1 \lambda_1^{n-3} \mu_1 + \dfrac{(n-2)(n-3)}{2!} B_2 \lambda_1^{n-4} \mu_1^2 + \cdots \\ \qquad \cdots + (n-2) B_{n-3} \lambda_1 \mu_1^{n-3} + B_{n-2} \mu_1^{n-2} = 0. \end{cases}$$

It is evident from the nature of the substitution S_1 that each of the coefficients B in equation (5) contains at least the first power of A_0. For the sake of simplicity we shall assume that this first power of A_0 has been excluded from (5).

We can easily verify the truth of the formula

$$B_r = (-1)^{r-1} \left[(r-1) A_1^r - \frac{r(r-1)}{2!} A_0 A_1^{r-1} A_2 + \frac{r(r-1)(r-2)}{3!} A_0^2 A_1^{r-2} A_3 - \cdots \right]$$

for determining any particular coefficient B. This formula, when read in reverse order, is

$$(6) \qquad \begin{cases} B_r = A_0^{r-1} A_r - r A_0^{r-2} A_1 A_{r-1} \\ + \dfrac{r(r-1)}{2!} A_0^{r-3} A_1^2 A_{r-2} - \dfrac{r(r-1)(r-2)}{3!} A_0^{r-4} A_1^3 A_{r-3} + \cdots . \end{cases}$$

Now introducing the notation

$$C_r = A_0 A_r - A_1 A_{r-1}$$

for all positive integral values of $r > 1$, the formula (6) can be written as

$$B_r = A_0^{r-2} C_r - (r-1) A_0^{r-3} A_1 C_{r-1} + \frac{(r-1)(r-2)}{2!} A_0^{r-4} A_1^2 C_{r-2} - \cdots .$$

This can easily be changed to

$$(7)\quad B_r = A_0^{r-2}C_r - (r-1)A_1B_{r-1} - \frac{(r-1)(r-2)}{2!}A_1^2B_{r-2} - \frac{(r-1)(r-2)(r-3)}{3!}A_1^3B_{r-3} - \cdots,$$

a formula that will greatly facilitate the removal of the u from the coefficients B.

From equations (6) and (7) we see that

$$B_0 = 1, \qquad B_1 = 0, \qquad B_2 = C_2 = A_0A_2 - A_1^2, \qquad B_3 = A_0C_3 - 2A_1B_2, \ldots,$$

from everyone of which it is possible to factor u_x to such a power that the u will no longer be found in the quotients. This will be proved below.

After having removed the u from the required number of coefficients B for any particular n-ic and having substituted the results in the discriminant of the equation (5) for that particular value of n, we shall have, by putting the discriminant equal to zero, the equation of a bitangential curve of any n-ic whatever.

§ 3.

Proof that the u can be removed from any coefficient B.

The removal of the u from the coefficients B is accomplished by the use of the identity

$$v_x(afu) = a_x(vfu) + f_x(avu) + u_x(afv)$$

and a number of modifications of it, which will be collected for reference in number (8).

If, in the equation

$$[\lambda a_x + \mu(afu)]^n = 0,$$

from which equation (2) was obtained, we put

$$\mu = \rho v_x,$$

we shall get the equation

$$[\lambda a_x + \rho v_x(afu)]^n = 0.$$

In this equation we can replace $v_x(afu)$ by the value given for it in the identity above, omitting the second term, since it contains $f_x = 0$. Making this substitution, we get the equation

$$\{[\lambda + \rho(vfu)]a_x + \rho u_x(afv)\}^n = 0,$$

which can be changed to the equation

$$[\lambda' a_x + \mu'(afv)]^n = 0,$$

by use of the substitution

$$(S')\qquad \begin{cases} \lambda' = \lambda + \rho(vfu) = \lambda + \dfrac{(vfu)}{v_x}\mu \\ \mu' = \quad \rho u_x \quad = \quad \dfrac{u_x}{v_x}\mu, \end{cases}$$

the determinant of which is

$$\begin{vmatrix} 1 & \dfrac{(vfu)}{v_x} \\ 0 & \dfrac{u_x}{v_x} \end{vmatrix} = \frac{u_x}{v_x}.$$

This new equation differs from the original in having v in the place of u. The operation of the substitution (S') is equivalent to having the point y take a new position on the tangent $f_y = 0$, determined by the intersection of the tangent and the new line $v_y = 0$. Since the line $u_y = 0$ was arbitrarily chosen in the first place, there can be no objection to this transformation.

We can now put upon the above equation in λ' and μ' the same conditions that were imposed upon the original equation in λ and μ to get equation (2). With these conditions assumed we get the equation

$$(2') \qquad \begin{cases} A'_0\lambda'^{n-2} + (n-2)A'_1\lambda'^{n-3}\mu' + \dfrac{(n-2)(n-3)}{2!}A'_2\lambda'^{n-4}\mu'^2 + \cdots \\ \qquad \cdots + (n-2)A'_{n-3}\lambda'\mu'^{n-3} + A'_{n-2}\mu'^{n-2} = 0, \end{cases}$$

where the coefficients A' differ from the coefficients A by having v in the place of u. Now operating on this equation the substitution

$$(S'_1) \qquad \begin{cases} \lambda' = \lambda'_1 - A'_1\mu'_1 \\ \mu' = \qquad A'_0\mu'_1, \end{cases}$$

the determinant of which is

$$\begin{vmatrix} 1 & -A'_1 \\ 0 & A'_0 \end{vmatrix} = A'_0,$$

we get the equation

$$(5') \qquad \begin{cases} B'_0\lambda'^{n-2}_1 + (n-2)B'_1\lambda'^{n-3}_1\mu'_1 + \dfrac{(n-2)(n-3)}{2!}B'_2\lambda'^{n-4}_1\mu'^2_1 + \cdots \\ \qquad \cdots + (n-2)B'_{n-3}\lambda'_1\mu'^{n-3}_1 + B'_{n-2}\mu'^{n-2}_1 = 0. \end{cases}$$

Here, the same as in equation (5), if we exclude from each of the coefficients B' the first power of A'_0, we shall have

$$B'_0 = 1, \qquad B'_1 = 0, \qquad B'_2 = A'_0A'_2 - A'^2_1, \ \ldots$$

the coefficients B' differing from the coefficients B by containing v in the place of u.

It is now possible to transform equation (5') into equation (5) by means of the composite substitution

$$(S_1^{-1}S'S_1) \qquad \begin{cases} \lambda'_1 = \lambda_1 + \dfrac{A_0A'_0(vfu) + A_0A'_1u_x - A'_0A_1v_x}{A'_0v_x}\mu_1 \\ \mu'_1 = \qquad\qquad \dfrac{A_0u_x}{A'_0v_x}\mu_1, \end{cases}$$

the determinant of which is

$$\begin{vmatrix} 1 & A_0\left[\dfrac{(vfu)}{v_x} + \dfrac{A'_1u_x}{A'_0v_x}\right] - A_1 \\ 0 & \dfrac{A_0u_x}{A'_0v_x} \end{vmatrix} = \frac{A_0u_x}{A'_0v_x}.$$

It will be shown in equation (17) below that A_0 is equal to a constant times the hessian of the n-ic times u_x^2. In the same way it can be shown that the coefficient A'_0 is equal to the same constant times the hessian of the n-ic times v_x^2. Therefore the determinant of the composite substitution which transforms equation (5′) into equation (5) is equal to

$$\frac{u_x^3}{v_x^3}.$$

Now, since the coefficients B and B' are seminvariants, posessing the property of invariants for substitutions such as the composite substitution above, we have the equation

$$B_r = \frac{u_x^{3r}}{v_x^{3r}} B'_r.$$

In this equation the subscript r is also the exponent of the power of the determinant involved in each pair of corresponding coefficients, as may be seen by referring to equation (7) and as will appear more clearly in the work below. We thus have the equation

$$\frac{B_r}{u_x^{3r}} = \frac{B'_r}{v_x^{3r}},$$

which proves that the u can be removed from the coefficients B separately.

§ 4.

An extraneous factor.

We thus have a method of developing the equation of a bitangential curve of the general plane n-ic curve which avoids the difficulty of having to remove the u from the discriminant of equation (2) as a whole. While in this particular we gain by the operation of the substitution S_1 on the equation (2), we thereby introduce an extraneous factor into the equation of the bitangential curve. The difficulty of its removal is small, however, when compared to the difficulty of the removal of the u from the discriminant of equation (2) without making use of the substitution S_1, as can be seen by comparing the work below for the quintic whith the work as given by MAISANO.

By the operation of the substitution S_1 on equation (2) in getting equation (5), the discriminant of equation (2) is made to differ from the discriminant of equation (5) by the $(n-2)(n-3)^{\text{th}}$ power of the determinant of the S_1, or A_0. But it will be remembered that we excluded an A_0 from each one of the coefficients B and therefore the discriminant of equation (5) will contain only the $(n-3)(n-4)^{\text{th}}$ power of the determinant A_0. It is thus seen that there is no extraneous factor in the equation of the bitangential curve of the general plane quartic curve gotten in this way. From the equation of the bitangential of the quintic thus gotten the square of the hessian of the quintic will have to be removed, from that of the sextic the sixth power of the

hessian will have to be removed, from that of the septimic the twelfth power and from that of the octic the twentieth power.

§ 5.

Formulae and covariants to be used.

We shall now develop some formulae for use in the removal of the u from the several coefficients B and for the removal of these extraneous factors. We shall encounter a large number of forms, the number increasing with the increasing value of n, and it is necessary that we adopt as short a notation for them as possible and collect them for reference.

The identity used in § 3 is the basis of all operations used in the various steps in removing the u and in removing the extraneous factors. It and a number of modifications of itself are here collected for reference. They are

$$(8)\quad \begin{cases} \alpha_x(abc) = a_x(\alpha bc) + b_x(\alpha ca) + c_x(\alpha ab) \;.\;.\;.\;.\;.\;.\;.\;A \\ (\alpha ab)(\alpha bc)a_x = \frac{1}{2}(\alpha ab)[a_x(\alpha bc) + b_x(\alpha ca)] \\ \qquad = \frac{1}{2}(\alpha ab)[\alpha_x(abc) - c_x(\alpha ab)] \;.\;.\;.\;.\;.\;.\;.\;B \\ (abc)(\alpha bc)a_x = \frac{1}{3}(abc)[a_x(\alpha bc) + b_x(\alpha ca) + c_x(\alpha ab)] \\ \qquad = \frac{1}{3}(abc)^2\alpha_x \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;C \\ (\alpha bc)(\alpha ad) + (\alpha ca)(\alpha bd) + (\alpha ab)(\alpha cd) = 0 \;.\;.\;.\;.\;.\;D \\ (\beta\gamma a)(\alpha bc) + (\beta\gamma b)(\alpha ca) + (\beta\gamma c)(\alpha ab) = (\alpha\beta\gamma)(abc) \;.\;.\;E. \end{cases}$$

For the purpose of abbreviation we shall put

$$a_x^{n-2} b_x^{n-2} (abu)^2 = u_\sigma^2$$
$$b_x^{n-2} c_x^{n-2} (abc)(bcu) = a_\sigma u_\sigma,$$

from which we get the formulae

$$(9)\quad \begin{cases} (a_\sigma u_\sigma)^2 = a_\sigma^2 u_\sigma^2 - \frac{2}{3}\Delta b_x^{n-2}(abu)^2 \\ (a_\sigma u_\sigma)(b_\sigma u_\sigma) = (a_\sigma b_\sigma)u_\sigma^2 - \frac{2}{3}\Delta c_x^{n-2}(acu)(bcu), \end{cases}$$

where we put

$$\Delta = (abc)^2 a_x^{n-2} b_x^{n-2} c_x^{n-2},$$

the Δ being the hessian of the n-ic, except for a constant factor, as will be shown in equation (11) below. We shall also put

$$\alpha_x^{3n-8} \beta_x^{3n-8} (\alpha\beta u)^2 = u_\omega^2$$
$$\alpha_x^{3n-8} \beta_x^{3n-8} (\alpha\beta f)(\alpha\beta u) = (f_\omega u_\omega).$$

By reference to equation (4) we have

$$a_y b_y = (afu)(bfu) = c_x^{n-2} d_x^{n-2} [c_x d_x (acu)(bdu)].$$

Interchanging c and d and adding the result to the above expression for $a_y b_y$ and dividing by 2, we have

$$a_y b_y = -\tfrac{1}{2} c_x^{n-2} d_x^{n-2}[-c_x d_x (a\,c\,u)(b\,d\,u) - c_x d_x (a\,d\,u)(b\,c\,u)].$$

Adding two terms to this second member, each of which contains $f_x = 0$, we have

$$\begin{aligned} a_y b_y &= -\tfrac{1}{2} c_x^{n-2} d_x^{n-2}[c_x^2 (adu)(bdu) - c_x d_x (acu)(bdu) - c_x d_x (adu)(bcu) + d_x^2 (acu)(bcu)] \\ &= -\tfrac{1}{2} c_x^{n-2} d_x^{n-2}[c_x (adu) - d_x (acu)][c_x (bdu) - d_x (bcu)] \\ &= -\tfrac{1}{2} c_x^{n-2} d_x^{n-2}[a_x (cdu) - (acd) u_x][b_x (cdu) - (bcd) u_x] \\ &= -\tfrac{1}{2}[a_x b_x u_\sigma^2 - a_x (b_\sigma u_\sigma) u_x - b_x (a_\sigma u_\sigma) u_x + (a_\sigma b_\sigma) u_x^2], \end{aligned}$$

first by factoring, then making use of $(8\,B)$ on each factor and finally multiplying the factors and using the abbreviations adopted above. If we replace b by a throughout this formula, we get another formula that will be frequently used. The two are written for reference as

$$(10) \qquad \begin{cases} a_y^2 = -\tfrac{1}{2}[a_x^2 u_\sigma^2 - 2\,a_x (a_\sigma u_\sigma) u_x + a_\sigma^2 u_x^2] \\ a_y b_y = -\tfrac{1}{2}[a_x b_x u_\sigma^2 - a_x (b_\sigma u_\sigma) u_x - b_x (a_\sigma u_\sigma) u_x + (a_\sigma b_\sigma) u_x^2]. \end{cases}$$

The formulae given in numbers (8), (9) and (10) will be used where the letters are quite different from those involved as they are written above, as in the case

$$(\Delta a f)(\Delta b f) = -\tfrac{1}{2}[a_x b_x \Delta_\sigma^2 - \Delta a_x (\Delta_\sigma b_\sigma) - \Delta b_x (\Delta_\sigma a_\sigma) + \Delta^2 (a_\sigma b_\sigma)].$$

For the hessian of the n-ic curve we have

$$\begin{aligned} \frac{1}{n^3 (n-1)^3} \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} &= a_x^{n-2} b_x^{n-2} c_x^{n-2} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} a_1 b_2 c_3 = a_x^{n-2} b_x^{n-2} c_x^{n-2} (abc) a_1 b_2 c_3 \\ &= \tfrac{1}{2} a_x^{n-2} b_x^{n-2} c_x^{n-2} (a\,b\,c)(b_2 c_3 - b_3 c_2) a_1 \\ &= \tfrac{1}{6} a_x^{n-2} b_x^{n-2} c_x^{n-2} (abc)[a_1 (b_2 c_3 - b_3 c_2) + b_1 (c_2 a_3 - c_3 a_2) + c_1 (a_2 b_3 - a_3 b_2)] \\ &= \tfrac{1}{6} a_x^{n-2} b_x^{n-2} c_x^{n-2} (a\,b\,c)^2 = \tfrac{1}{6} a_x^{n-2} a_\sigma^2 = \tfrac{1}{6} \Delta. \end{aligned}$$

For reference we shall put

$$(11) \qquad a_x^{n-2} b_x^{n-2} c_x^{n-2} (a\,b\,c)^2 = a_x^{n-2} a_\sigma^2 = \alpha_x^{3n-6} = \beta_x^{3n-6} = \gamma_x^{3n-6} = \cdots = \Delta,$$

with the understanding that Δ and the letters α, β, γ, δ will be reserved for the representation of the hessian of the n-ic, the letter Δ to be used in the same relation to the others that f is used in relation to the letters a, b, c, d, etc., for the n-ic.

In the development of a symbolic representation for the covariant

$$\Theta = \begin{vmatrix} f_{11} & f_{12} & f_{13} & \Delta_1 \\ f_{21} & f_{22} & f_{23} & \Delta_2 \\ f_{31} & f_{32} & f_{33} & \Delta_3 \\ \Delta_1 & \Delta_2 & \Delta_3 & 0 \end{vmatrix} = \left[\begin{matrix} \Delta \\ \Delta \end{matrix}\right],$$

it will be necessary to supply the numerical coefficient $n(n-1)$ which belongs with

each f_{ij} and the coefficient $\frac{1}{2}n^3(n-1)^3(n-2)$ which belongs with each Δ_i. We thus have

$$\frac{-4}{n^8(n-1)^8(n-2)^2}\Theta = a_x^{n-2}b_x^{n-2}(\Delta ab)(\Delta_1 a_2 b_3 - \Delta_2 a_1 b_3 + \Delta_3 a_1 b_2)$$
$$= \tfrac{1}{2}a_x^{n-2}b_x^{n-2}(\Delta ab)[\Delta_1(a_2 b_3 - a_3 b_2) - \Delta_2(a_1 b_3 - a_3 b_1) + \Delta_3(a_1 b_2 - a_2 b_1)]$$
$$= \tfrac{1}{2}a_x^{n-2}b_x^{n-2}(\Delta ab)^2 = \tfrac{1}{2}\Delta_\sigma^2 = \tfrac{1}{2}\theta$$

whence we can write

$$(12) \qquad -\frac{8}{n^8(n-1)^8(n-2)^2}\Theta = a_x^{n-2}b_x^{n-2}(\Delta ab)^2 = \Delta_\sigma^2 = \theta.$$

The symbolic representation for the covariant

$$\Phi = \begin{vmatrix} f_{11} & f_{12} & f_{13} & \frac{\partial}{\partial x_1} \\ f_{21} & f_{22} & f_{23} & \frac{\partial}{\partial x_2} \\ f_{31} & f_{32} & f_{33} & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 \end{vmatrix} \Delta = \begin{bmatrix} \partial_x \\ \partial_x \end{bmatrix} \Delta$$

can be gotten from the equation (12) by replacing

$$\frac{n^6(n-1)^6(n-2)^2}{4}\Delta_1^2 \quad \text{with} \quad \frac{n^3(n-1)^3(n-2)(3n-7)}{2}\Delta_{11},$$

$$\frac{n^6(n-1)^6(n-2)^2}{4}\Delta_1\Delta_2 \quad \text{with} \quad \frac{n^3(n-1)^3(n-2)(3n-7)}{2}\Delta_{12},$$

and so on for all the combinations of the first derivatives. We thus get

$$(13) \qquad \begin{cases} -\dfrac{4}{n^3(n-1)^3(n-2)(3n-7)}\Phi = \alpha_x^{3n-8}a_x^{n-2}b_x^{n-2}(\alpha ab)^2 \\ \qquad = \alpha_x^{3n-8}\alpha_\sigma^2 = \varphi' = \dfrac{n-3}{3n-7}\varphi, \end{cases}$$

where we have put

$$\varphi = a_x^{n-4}(a_\sigma^2)^2 = a_x^{n-4}b_x^{n-2}c_x^{n-2}d_x^{n-2}e_x^{n-2}(abc)^2(ade)^2.$$

Taking the first polar of the hessian of the n-ic

$$\Delta = \alpha_x^{3n-6} = (abc)^2 a_x^{n-2}b_x^{n-2}c_x^{n-2} = a_x^{n-2}a_\sigma^2,$$

with respect to the point y, we get

$$\Delta_y = \alpha_x^{3n-7}\alpha_y = (abc)^2 a_x^{n-3}b_x^{n-2}c_x^{n-2}a_y = a_x^{n-3}a_\sigma^2 a_y.$$

Here the subscripts y may be replaced by determinants of the second order and the formula thus gotten be used for purposes of abbreviation. For instance we may put

$$(\Delta_\sigma f_\sigma) = a_x^{n-3}a_\sigma^2(a_\sigma f_\sigma).$$

Taking the second polar of the hessian with respect to the point y, we get

$$(3n-7)\alpha_x^{3n-8}\alpha_y^2 = (n-3)a_x^{n-4}a_\sigma^2 a_y^2 + 2(n-2)(abc)^2 a_x^{n-3}b_x^{n-3}c_x^{n-2}a_y b_y,$$

from which, by making the substitution

$$y_i = d_x^{n-2}e_x^{n-2}(de)_i,$$

we get the equation

$$(3n-7)\alpha_x^{3n-8}\alpha_\sigma^2 = (n-3)a_x^{n-4}(a_\sigma^2)^2 + 2(n-2)(abc)^2 a_x^{n-3}b_x^{n-3}c_x^{n-2}(a_\sigma b_\sigma),$$

where the last term contains $f_x = 0$, as we shall prove immediately. With the proof of this fact we shall have established the truth of the last equality in equation (13).

To prove that the term

$$(abc)^2(ade)(bde)a_x^{n-3}b_x^{n-3}c_x^{n-2}d_x^{n-2}e_x^{n-2}$$

contains the factor f_x, we start with the equation

$$(abc)(bcd)(ade)[b_x(cad)+c_x(abd)]a_x^{n-3}b_x^{n-3}c_x^{n-3}d_x^{n-3}e_x^{n-1} = 0,$$

which is seen to be true by interchanging b and c in either of the terms. In the first term we replace $b_x(ade)$, by the use of $(8A)$, and we get the equation

$$\begin{aligned}(abc)(bcd)(cad)[a_x(bde) + d_x(abe) - e_x(abd)]a_x^{n-3}b_x^{n-3}c_x^{n-3}d_x^{n-3}e_x^{n-1}\\ + (abc)(bcd)(ade)(abd)a_x^{n-3}b_x^{n-3}c_x^{n-2}d_x^{n-3}e_x^{n-1} = 0,\end{aligned}$$

the third term of which contains f_x and can therefore be neglected. In the first term interchange a and d and in the last term interchange first c and d and then b and c. The two terms thus gotten for the first and last terms are alike and they are also like the second term. We thus have the equation

$$3(abc)(bcd)(abe)(cad)a_x^{n-3}b_x^{n-3}c_x^{n-3}d_x^{n-2}e_x^{n-1} = 0.$$

Now, by the use of $(8A)$, we replace $e_x(abc)$ and get the equation

$$3(bcd)(abe)(cad)[a_x(bce) + b_x(cae) + c_x(abe)]a_x^{n-3}b_x^{n-3}c_x^{n-3}d_x^{n-2}e_x^{n-2} = 0,$$

the last term of which is the form desired and the first two terms of which are alike if we interchange a and b in either of them. Taking the first term and adding to itself what it becomes by interchanging a and d, we get the equivalent of the first two terms. They are

$$3(bcd)(cad)[(abe)(bcd) + (abc)(bde)]a_x^{n-2}b_x^{n-3}c_x^{n-3}d_x^{n-2}e_x^{n-2},$$

which, by use of $(8D)$, becomes

$$3(bce)^2(cad)(abd)a_x^{n-2}b_x^{n-3}c_x^{n-3}d_x^{n-2}e_x^{n-2},$$

an expression like the term sought by properly interchanging the letters. We have then

as a result

$$(abc)^2(a_\sigma b_\sigma)a_x^{n-3}b_x^{n-3}c_x^{n-2} = \tfrac{1}{6}(abc)(bcd)(abd)(acd)a_x^{n-3}b_x^{n-3}c_x^{n-3}d_x^{n-3}e_x^n = 0,$$

and the last equality in number (13) is established.

We shall frequently find the covariant gotten by bordering the matrix of the hessian of the hessian of the n-ic with the first derivatives of the n-ic. It is the form gotten by interchanging f and Δ in the determinant expression for Θ and is put in the symbolic form as follows:

$$\frac{-4}{n^8(n-1)^8(n-2)^2(3n-7)^2}\begin{vmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & f_1 \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & f_2 \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & f_3 \\ f_1 & f_2 & f_3 & 0 \end{vmatrix}$$

$$= \alpha_x^{3n-8}\beta_x^{3n-8}(\alpha\beta f)(f_1\alpha_2\beta_3 - f_2\alpha_1\beta_3 + f_3\alpha_1\beta_2)$$

$$= \tfrac{1}{2}\alpha_x^{3n-8}\beta_x^{3n-8}(\alpha\beta f)[f_1(\alpha_2\beta_3 - \alpha_3\beta_2) - f_2(\alpha_1\beta_3 - \alpha_3\beta_2) + f_3(\alpha_1\beta_2 - \alpha_2\beta_1)]$$

$$= \tfrac{1}{2}\alpha_x^{3n-8}\beta_x^{3n-8}(\alpha\beta f)^2 = \tfrac{1}{2}f_\omega^2.$$

By equations (10), (12) and (13), we have

$$f_\omega^2 = \theta - \Delta\varphi'. \tag{14}$$

If, in the first member of equation (14), we replace f_1^2 with f_{11}, replace $f_1 f_2$ with f_{12} and so on throughout the expression for the square of the determinant, we shall get a symbolic representation for the covariant

$$\Psi = \begin{vmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & \dfrac{\partial}{\partial x_1} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & \dfrac{\partial}{\partial x_2} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} & 0 \end{vmatrix} f.$$

Making these substitutions, remembering to supply the proper numerical coefficients, we get

$$\frac{-8}{n^7(n-1)^7(n-2)^2(3n-7)^2}\Psi = \alpha_x^{3n-8}\beta_x^{3n-8}a_x^{n-2}(\alpha\beta a)^2 = a_x^{n-2}a_\omega^2 = \psi. \tag{15}$$

In addition to the covariants given in equations (11), (12), (13), (14) and (15), we shall give in number (16) a list of abbreviations for the forms that will appear in the work following. The forms given at the last of the list, represented by the letters Σ are those that appear only in the case of the octic. The equation of the bitangential of the octic contains most of the other forms also. Those represented by the letters Π make their appearance first in the case of the septimic. The form represented by Λ is the only one appearing in the case of the sextic which does not also occur in the case of the quintic. Some of these forms may possibly contain $f_x = 0$.

(16)
$$\left\{\begin{array}{ll}
\Delta_\Delta = \alpha_x^{3n-8}\alpha_\omega^2 = \alpha_x^{3n-8}\beta_x^{3n-8}\gamma_x^{3n-8}(\alpha\beta\gamma)^2, & \Gamma_1 = a_x^{n-5}(a_\sigma^2)^2(\Delta_\sigma a_\sigma),\\
\xi = a_x^{n-5}(a_\sigma^2)^2(\Delta af), \quad \varepsilon = a_x^{n-6}(a_\sigma^2)^3, & \Gamma_2 = \alpha_x^{3n-8}a_x^{n-3}b_x^{n-2}(\Delta_\sigma a_\sigma)(\alpha ab)^2,\\
\eta = a_x^{n-3}b_x^{n-2}(\Delta ab)^2(\Delta af), & \Pi_1 = a_x^{n-7}(a_\sigma^2)^3(\Delta af),\\
\zeta = a_x^{n-4}a_\sigma^2(\Delta_\sigma a_\sigma)(\Delta af), & \Pi_2 = a_x^{n-5}b_x^{n-2}a_\sigma^2(\Delta ab)^2(\Delta af),\\
\Upsilon_1 = a_x^{n-5}b_x^{n-5}(a_\sigma^2)^2(b_\sigma^2)^2(a_\sigma b_\sigma), & \Pi_3 = a_x^{n-6}(a_\sigma^2)^2(\Delta_\sigma a_\sigma)(\Delta af),\\
\Upsilon_2 = a_x^{n-5}b_x^{n-4}c_x^{n-2}(a_\sigma^2)^2b_\sigma^2(abc)(\Delta bc), & \Pi_4 = \alpha_x^{3n-8}a_x^{n-3}b_x^{n-2}(\alpha ab)^2(\Delta af),\\
\Omega_1 = \alpha_x^{3n-8}a_x^{n-4}b_x^{n-2}a_\sigma^2(\alpha ab)^2, & \Pi_5 = \alpha_x^{3n-8}a_x^{n-4}a_\sigma^2(\alpha_\sigma a_\sigma)(\alpha af),\\
\Omega_2 = a_x^{n-5}b_x^{n-3}c_x^{n-2}(a_\sigma^2)^2(a_\sigma b_\sigma)(\Delta bc)^2, & \Pi_6 = a_x^{n-4}b_x^{n-2}(\Delta_\sigma a_\sigma)(\Delta ab)^2(\Delta af),\\
\Omega_3 = \alpha_x^{3n-8}a_x^{n-3}b_x^{n-3}c_x^{n-2}d_x^{n-2}(a_\sigma b_\sigma)(\alpha bd)^2(\Delta ac)^2, & \Sigma_1 = a_x^{n-8}(a_\sigma^2)^4,\\
\Omega_4 = a_x^{n-3}b_x^{n-2}(a_\omega f_\omega)(\Delta ab)^2, & \Sigma_2 = a_x^{n-7}(a_\sigma^2)^3(\Delta_\sigma a_\sigma),\\
\Omega_5 = a_x^{n-4}b_x^{n-3}c_x^{n-2}d_x^{n-2}a_\sigma^2(abd)(\Delta ad)(\Delta bc)^2, & \Sigma_3 = a_x^{n-6}b_x^{n-2}(a_\sigma^2)^2(\Delta ab),\\
\Omega_6 = a_x^{n-4}b_x^{n-4}c_x^{n-2}a_\sigma^2b_\sigma^2(\Delta_\sigma a_\sigma)(abc)(\Delta bc), & \Sigma_4 = a_x^{n-4}b_x^{n-2}c_x^{n-2}(\Delta ab)^2(\Delta ac)^2,\\
\Lambda = a_x^{n-3}a_\omega^2(\Delta_\sigma a_\sigma), & \Sigma_5 = a_x^{n-5}b_x^{n-2}a_\sigma^2(\Delta_\sigma a_\sigma)(\Delta ab)^2.
\end{array}\right.$$

§ 6.

Reduction of the expressions for A_0, A_1 and C_r.

By reference to the form for B_r given in equation (7), we see that we must reduce the expressions for A_0, A_1 and C_r so that the u contained in each will be given explicitly in terms of u_x as far as possible. From equations (3) and (10) we have

$$A_0 = \frac{n(n-1)}{2}a_x^{n-2}a_y^2 = -\frac{n(n-1)}{4}a_x^{n-2}[a_x^2u_\sigma^2 - 2a_x(a_\sigma u_\sigma)u_x + a_\sigma^2u_x^2],$$

the first term of which contains $f_x = 0$ and the second of which, by (8 C), is equal to

$$-\tfrac{2}{3}a_x^{n-2}a_\sigma^2u_x^2.$$

We thus have, by (11),

(17)
$$A_0 = -\frac{n(n-1)}{2^2\cdot 3}a_x^{n-2}a_\sigma^2u_x^2 = -\frac{n(n-1)}{2^2\cdot 3}\Delta u_x^2.$$

Similarly, we have

$$A_1 = -\frac{n(n-1)}{2^2\cdot 3}a_x^{n-3}a_y[a_x^2u_\sigma^2 - 2a_x(a_\sigma u_\sigma)u_x + a_\sigma^2u_x^2],$$

the first term of which contains $f_y = 0$ and the second of which, by (8 C), contains $u_y = 0$. We thus have

(18)
$$A_1 = -\frac{n(n-1)}{2^2\cdot 3}a_x^{n-3}a_\sigma^2a_yu_x^2 = -\frac{n(n-1)}{2^2\cdot 3}\Delta_yu_x^2.$$

From equations (3), (17) and (18) we get

$$C_r = A_0 A_r - A_1 A_{r-1} = -\frac{n^2(n-1)^2}{2^2 \cdot 3\, r(r+1)(r+2)} a_x^{n-r-2} a_y^{r+1} [r\Delta a_y + (r+2) a_x \Delta_y],$$

the development of which will be made in two parts, one with r an even number and the other with r taken as an odd number. Supposing that $r = 2t$, we have

$$(19) \quad \left\{ \begin{array}{c} C_{2t} = \frac{(-1)^{t+1} n^2 (n-1)^2}{2^{t+4} \cdot 3\, t(t+1)(2t+1)} a_x^{n-2t-2} [a_x^2 u_\sigma^2 - 2a_x (a_\sigma u_\sigma) u_x + a_\sigma^2 u_x^2]^t \times \\ \times [\Delta a_x^2 u_\sigma^2 + (t-1)\Delta a_x (a_\sigma u_\sigma) u_x - (t+1) a_x^2 (\Delta_\sigma u_\sigma) u_x - t\Delta a_\sigma^2 u_x^2 + (t+1) a_x (\Delta_\sigma a_\sigma) u_x^2] u_x^2; \end{array} \right.$$

and supposing that $r = 2t - 1$, we have

$$(20) \quad \left\{ \begin{array}{c} C_{2t-1} = \frac{(-1)^{t+1} n^2 (n-1)^2}{2^{t+3} \cdot 3\, t(2t-1)(2t+1)} a_x^{n-2t-1} [a_x^2 u_\sigma^2 - 2a_x (a_\sigma u_\sigma) u_x + a_\sigma^2 u_x^2]^t \times \\ \times [(2t-1)\Delta a_y - (2t+1) a_x \Delta_y] u_x^2. \end{array} \right.$$

Since C_r is equal to zero for $r = 1$ and has no meaning for $r < 1$, we shall restrict t to positive integral values not less than 1 in equation (19) and not less than 2 in equation (20).

Making $t = 1$ in equation (19), multiplying and omitting the two terms in the product which contain $f_x = 0$ and the two terms that are alike and opposite in sign, we have

$$C_2 = \frac{n^2(n-1)^2}{2^6 \cdot 3^2} [-2\Delta a_x^{n-1} (a_\sigma u_\sigma) u_\sigma^2 u_x + 2a_x^{n-1} (\Delta_\sigma a_\sigma) u_\sigma^2 u_x^2 + 4a_x^{n-1} (a_\sigma u_\sigma)(\Delta_\sigma u_\sigma) u_x^2$$
$$- 2a_x^{n-2} a_\sigma^2 (\Delta_\sigma u_\sigma) u_x^2 + 2\Delta a_x^{n-3} a_\sigma^2 (a_\sigma u_\sigma) u_x^3 - 4a_x^{n-2} (\Delta_\sigma a_\sigma)(a_\sigma u_\sigma) u_x^3 - \Delta\varphi\, u_x^4 + 2\theta\, u_x^4] u_x^2.$$

In this expression for C_2 the first and second terms cancel, by using (8 C) on each term, the third and sixth terms cancel, by using (8 C) on each term, the fourth and fifth terms cancel and we have

$$(21) \qquad B_2 = C_2 = \frac{n^2(n-1)^2}{2^6 \cdot 3^2} [2\theta - \Delta\varphi] u_x^6 = \frac{n^2(n-1)^2}{2^6 \cdot 3^2} H u_x^6,$$

where we have put

$$H = 2\theta - \Delta\varphi.$$

Now making $n = 4$ and equating the discriminant of equation (5) to zero, we have for the equation of a bitangential curve of the general plane quartic curve the equation

$$H = 0,$$

from which, by use of the equations (11), (12) and (13), we get the well known form

$$\Theta - 3\Delta\Phi = 0,$$

as given by SALMON in « *Higher Plane Curves* ».

§ 7.

The equation of a bitangential of the quintic.

Proceeding to the development of the equation of a bitangential of the quintic, we must first find an expression for B_3 with the u removed from it. From equation (7) we have

$$B_3 = A_0 C_3 - 2 A_1 B_2,$$

where the value of C_3 must be obtained from equation (20) by making $t = 2$, squaring and multiplying as there indicated. Doing this and making use of equation (9) in one of the terms and cancelling, we have

$$A_0 C_3 = \frac{n^3(n-1)^3}{2^8 \cdot 3^3 \cdot 5}[8\Delta^2\Delta_y u_\sigma^2 - 8\Delta^3 a_x^{n-3} b_x^{n-2}(abu)^2 a_y + 20\Delta(\Delta_\sigma u_\sigma)\Delta_y u_x$$
$$- 12\Delta^2 a_x^{n-4} a_\sigma^2 (a_\sigma u_\sigma) a_y u_x + 3\Delta^2 a_x^{n-5}(a_\sigma^2)^2 a_y u_x^2 - 5\Delta\varphi\Delta_y u_x^2] u_x^6.$$

The first two terms within the braces in this expression for $A_0 C_3$ are equal to

$$8\Delta^2\Delta_y u_\sigma^2 - 8\Delta^2 a_x^{n-3} b_x^{n-2}[a_x\Delta_y + (\Delta af)u_x](abu)^2,$$

the first two terms of which cancel. Denoting the other term by T, for convenience, we have

$$T = -8\Delta a_x^{n-3} b_x^{n-2}(\Delta af)[(\Delta ab)u_x + a_x(\Delta bu) - b_x(\Delta au)](abu)u_x = T_1 + T_2 + T_3.$$

Proceeding with these terms separately, we have

$$T_1 = -8\Delta a_x^{n-3} b_x^{n-2}(\Delta ab)[(\Delta ab)a_y + (abf)(\Delta au)]u_x^2,$$

the first term of which will be left unchanged and the second term of which is equal to

$$-8 a_x^{n-3} b_x^{n-2}(\Delta ab)[a_x(\Delta bf) - b_x(\Delta af)](\Delta au)u_x^2,$$

which in turn is equal to

$$4\theta\Delta_y u_x^2 - 4 a_x^{n-3}[a_x^2\theta - 2\Delta a_x(\Delta_\sigma a_\sigma) + \Delta^2 a_\sigma^2](\Delta au)u_x^2.$$

Here the first two terms cancel and the last two contain $(\Delta\Delta u) = 0$, whence we have

$$T_1 = -8\Delta a_x^{n-3} b_x^{n-2}(\Delta ab)^2 a_y u_x^2.$$

For the second term in T above we have

$$T_2 = -8\Delta a_x^{n-2} b_x^{n-2}(\Delta af)(\Delta bu)(abu)u_x$$
$$= -4\Delta a_x^{n-2} b_x^{n-2}[(\Delta\Delta f)(abu) + (\Delta ab)\Delta_y](abu)u_x,$$

the first term of which is zero, thus making

$$T_2 = -4\Delta(\Delta_\sigma u_\sigma)\Delta_y u_x.$$

For the third term in T we have

$$T_3 = 8\Delta a_x^{n-3}(\Delta af)(\Delta au)a_y u_x$$
$$= -4\Delta a_x^{n-3}[a_x^2(\Delta_\sigma u_\sigma) - \Delta a_x(a_\sigma u_\sigma) - a_x(\Delta_\sigma a_\sigma)u_x + \Delta a_\sigma^2 u_x](\Delta au)u_x,$$

the second term of which contains $(\Delta u u) = 0$ and each of the last two terms of which contains $(\Delta\Delta u) = 0$. We thus have

$$T_3 = -4\Delta(\Delta_\sigma u_\sigma)\Delta_y u_x .$$

Collecting, we get for the first two terms in $A_0 C_3$,

$$T = -8\Delta(\Delta_\sigma u_\sigma)\Delta_y u_x - 8\Delta a_x^{n-3} b_x^{n-2}(\Delta a b)^2 a_y u_x^2 .$$

The fourth term in $A_0 C_3$ is equal to

$$-12\Delta a_x^{n-4} a_\sigma^2 [a_x \Delta_y + (\Delta a f) u_x](a_\sigma u_\sigma) u_x$$
$$= -12\Delta(\Delta_\sigma u_\sigma)\Delta_y u_x - 12\Delta a_x^{n-4} a_\sigma^2 a_\sigma [\Delta_\sigma a_y - \varphi_\sigma \Delta_y + f_\sigma(\Delta a u)] u_x^2 ,$$

the last term of which contains $(\Delta\Delta u) = 0$. Adding the other three terms to the terms gotten for T, and replacing the first, second and fourth terms in $A_0 C_3$ with their sum, we get

$$A_0 C_3 = \frac{n^3(n-1)^3}{2^8 \cdot 3^3 \cdot 5} \left\{ \begin{array}{l} 3\Delta a_x^{n-5}(a_\sigma^2)^2 - 8 a_x^{n-3} b_x^{n-2}(\Delta a b)^2 \\ -12 a_x^{n-4} a_\sigma^2 (\Delta_\sigma a_\sigma) + 7\varphi a_x^{n-3} a_\sigma^2 \end{array} \right\} \Delta a_y u_x^6 .$$

Now putting

$$\Delta a_y = a_x \Delta_y + (\Delta a f) u_x$$

and multiplying, we have

$$A_0 C_3 = \frac{n^3(n-1)^3}{2^8 \cdot 3^3 \cdot 5} (G u_x - 10 H \Delta_y) u_x^8 , \tag{22}$$

where we have put

$$\left\{ \begin{array}{l} G = \Delta G_0 + G_1 \\ G_0 = 3\xi \\ G_1 = -8\eta - 12\zeta . \end{array} \right. \tag{23}$$

From equations (18) and (21) we have for the second term in B_3 the equation

$$-2 A_1 B_2 = \frac{n^3(n-1)^3}{2^7 \cdot 3^3} H \Delta_y u_x^8 .$$

Now making substitutions for the two terms in B_3, we get

$$B_3 = \frac{n^3(n-1)^3}{2^8 \cdot 3^3 \cdot 5} G u_x^9 . \tag{24}$$

Putting $n = 5$ in equation (5) and equating the discriminant to zero, we have

$$B_3^2 + 4 B_2^3 = 0 .$$

Substituting for B_2 and B_3 the forms given for them in equations (21) and (24), we have for the equation of a bitangential curve of the general plane quintic curve the equation

$$G^2 + 25 H^3 = 0 . \tag{25}$$

§ 8.

Removal of the extraneous factor Δ^2.

In order to remove the extraneous factor Δ^2 from the equation of the bitangential of the quintic as given in equation (25), we have from equation (23) the equation

$$G^2 = (\Delta G_0 + G_1)^2 = \Delta^2 G_0^2 + 2\Delta G_0 G_1 + G_1^2 ,$$

the three terms of which will have to be reduced. From equations (16) we have

$$\begin{aligned} G_0^2 = 9\xi^2 &= 9\, a_x^{n-5} b_x^{n-5} (a_\sigma^2)^2 (b_\sigma^2)^2 (\Delta af)(\Delta bf) \\ &= 9\, \alpha_x^{3n-8} a_x^{n-5} b_x^{n-5} (a_\sigma^2)^2 (b_\sigma^2)^2 (\alpha af)[\Delta(\alpha bf) + b_x(\Delta \alpha f)] \\ &= -\tfrac{9}{2} \Delta\, \alpha_x^{3n-8} a_x^{n-5} b_x^{n-5} (a_\sigma^2)^2 (b_\sigma^2)^2 [\alpha_x^2 (a_\sigma b_\sigma) - 2\alpha_x b_x (\alpha_\sigma a_\sigma) + a_x b_x \alpha_\sigma^2] \\ &\quad - \tfrac{9}{2} \varphi\, \alpha_x^{3n-8} \beta_x^{3n-8} a_x^{n-5} (a_\sigma^2)^2 (\alpha\beta f)[a_x(\alpha\beta f) - f_x(\alpha\beta a)], \end{aligned}$$

$$\Delta^2 G_0^2 = -\tfrac{9}{2} \Delta^2 \varphi^2 \theta + 9\Delta^3 \varphi \Gamma_1 - \tfrac{9}{2} \Delta^4 \Upsilon_1 .$$

Proceeding in the same way with the other terms in G^2 and adding results, we get

$$(26) \qquad G^2 = \left\{ \begin{array}{l} -200\theta^3 + 300\Delta\varphi\theta^2 - \frac{225}{2}\Delta^2\varphi^2\theta \\ -20\Delta^2\theta\Gamma + 15\Delta^3\varphi\Gamma + 8\Delta^3\Omega + \Delta^4\Upsilon \end{array} \right\},$$

where we have put

$$(27) \qquad \left\{ \begin{array}{l} \Gamma = 3\Gamma_1 - 8\Gamma_2 \\ \Upsilon = 72\varphi\Omega_1 - \frac{9}{2}\Upsilon_1 - 24\Upsilon_2 - 16\varphi\psi \\ \Omega = \frac{22}{3}\theta\psi - 20\theta\Omega_1 + 3\Omega_2 - 4\Omega_3 - \frac{4}{3}\Omega_4 - 8\Omega_5 + 6\Omega_6 . \end{array} \right.$$

From equation (21) we get

$$25H^3 = 200\theta^3 - 300\Delta\varphi\theta^2 + 150\Delta^2\varphi^2\theta - 25\Delta^3\varphi^3 ,$$

the first two terms of which are cancelled by the first two terms of the expression for G^2, given in equation (26), when the substitutions are made in equation (25). We thus have

$$(28) \qquad G^2 + 25H^3 = \Delta^2 \left\{ \begin{array}{l} \frac{75}{2}\varphi^2\theta - 5(4\theta - 3\Delta\varphi)\Gamma \\ -25\Delta\varphi^3 + 8\Delta\Omega + \Delta^2\Upsilon \end{array} \right\} = 0$$

for the equation of a bitangential curve of the general plane quintic curve with the extraneous factor Δ^2 removed.

In one of the methods given by SALMON in « *Higher Plane Curves* » there is an extraneous factor involved, but it is not a function of the hessian of the *n*-ic as in this case. In the equation of the bitangential of the quintic given in HEAL's first paper there occurs the square of the hessian of the quintic as an extraneous factor. It will be of interest here to sketch his method and prove the formula which he used in the removal of the Δ^2 in his second paper.

He uses the equation of the quintic in the form having the x-axis tangent to the curve at the origin. Making y equal to zero in this quintic, he shows that the discriminant of the cubic in x thus obtained is a factor of the absolute term in a certain equation which possesses the proper orders for a bitangential as suggested by SALMON. This equation is

$$(29) \quad [297\Delta J(f, \Delta, \Phi) - 456\Delta J'(f, \Delta, \Phi) - 20J(f, \Delta, \Theta)]^2 = 3025(4\Theta - 9\Delta\Phi)^3,$$

in which $J(f, \Delta, \Phi)$ is the jacobian of the forms f, Δ, Φ and $J'(f, \Delta, \Phi)$ means that

in forming the jacobian the Φ is to be differentiated on the assumption that the second derivatives of Δ which enter Φ are constants. If the discriminant of the cubic in x, gotten by making y equal to zero in the quintic, be equal to zero, then the origin is a point of bitangency and the absolute term of the equation (29) is zero. Thus, with this assumption, equation (29) becomes a bitangential of the quintic whose coefficients satisfy the assumption. The quintic in this case is not the general plane quintic in the full sense as treated above.

In Mr. HEAL's second paper he develops equation (29) by use of the formula for the product of two jacobians and gets a form which contains Δ^2 in every term except two. These two terms are

$$\Delta\Theta\left\{2\Phi\Theta - \begin{bmatrix}\Theta\\ \Delta\end{bmatrix}\right\},$$

where, after the notation of CLEBSCH, the covariant $\begin{bmatrix}\Theta\\ \Delta\end{bmatrix}$ is the matrix of the hessian of the n-ic bordered on one side by the first derivatives of Θ and on the other by the first derivatives of Δ. To remove the Δ^2 from these two terms he uses the formula

$$64\left\{\begin{bmatrix}\Theta\\ \Delta\end{bmatrix} - 2\Phi\Theta\right\} = 171\,\Delta^2\Psi + 72\,\Delta\begin{bmatrix}\Phi'\\ \Delta\end{bmatrix}. \tag{30}$$

Some letters concerning the proof of this formula passed between CAYLEY and HEAL. They can be seen in the « Proceedings of the London Mathematical Society », in « The Annals of Mathematics » and in CAYLEY's Collected Works.

To prove this formula we first find symbolic representations for the covariants involved which we have not already used. In the same way that we developed Θ, we get

$$\begin{bmatrix}\Phi'\\ \Delta\end{bmatrix} = -\frac{n^2(n-1)^2}{2}(\Phi'_\sigma\Delta_\sigma),$$

where the Φ' was defined in giving the meaning of J' in equation (29). Supplying the numerical coefficients in these derivatives we have

$$\frac{2^3}{n^{10}(n-1)^{10}(n-2)^3(3n-7)}\begin{bmatrix}\Phi'\\ \Delta\end{bmatrix} = \alpha_x^{3n-8}a_x^{n-3}b_x^{n-2}(\Delta_\sigma a_\sigma)(\alpha a b)^2 = \Gamma_2.$$

Similarly, we have

$$\frac{2^4}{n^{13}.(n-1)^{13}(n-2)^3}\begin{bmatrix}\Theta\\ \Delta\end{bmatrix} = (3n-7)\alpha_x^{3n-8}(\alpha_\sigma\Delta_\sigma)^2 + (n-2)a_x^{n-3}b_x^{n-2}(\Delta_\sigma a_\sigma)(\Delta a b)^2.$$

In this last expression we have for the first term

$$\alpha_x^{3n-8}(\alpha_\sigma\Delta_\sigma)^2 = \varphi'\theta - \tfrac{1}{3}\Delta^2\psi,$$

and for the second term

$$a_x^{n-3}b_x^{n-2}(\Delta_\sigma a_\sigma)(\Delta a b)^2 = \Delta\Gamma_2 - \tfrac{1}{6}\Delta^2\psi.$$

We thus have

$$\frac{2^4}{n^{13}(n-1)^{13}(n-2)^3}\begin{bmatrix}\Theta\\ \Delta\end{bmatrix} = (3n-7)\varphi'\theta + (n-2)\Delta\Gamma_2 - \frac{7n-16}{6}\Delta^2\psi,$$

which, by the use of equations (11), (12), (13) and (15), becomes

$$\begin{bmatrix}\Theta\\ \Delta\end{bmatrix} = 2\Phi\Theta + \frac{3(n-2)}{3n-7}\Delta\begin{bmatrix}\Phi'\\ \Delta\end{bmatrix} + \frac{3(n-2)(7n-16)}{(3n-7)^2}\Delta^2\Psi.$$

Making $n = 5$, we get HEAL'S formula as given in equation (30).

§ 9.

The equation of a bitangential of the sextic.

To find the equation of a bitangential of the sextic we must first find an expression for B_4 which has the u removed from it. From equation (7) we have

$$B_4 = A_0^2 C_4 - 3A_1 B_3 - 3A_1^2 B_2.$$

To get an expression for the first term in the second member of this equation, we put $t = 2$ in equation (19), square and multiply as indicated in (19), make use of equation (9), cancel and collect terms. We get

$$\begin{aligned} A_0^2 C_4 = \frac{n^4(n-1)^4}{2^{11}\cdot 3^4\cdot 5}\Big[&-15\Delta^2 H u_\sigma^2 u_x + \tfrac{8}{3}\Delta^4 a_x^{n-3} b_x^{n-2}(a_\sigma u_\sigma)(abu)^2 - 8\Delta^4 a_x^{n-4} b_x^{n-2} a_\sigma^2 (abu)^2 u_x \\ &+ 8\Delta^3 a_x^{n-3} b_x^{n-2}(\Delta_\sigma a_\sigma)(abu)^2 u_x + 8\Delta^3 a_x^{n-2}(\Delta au)^2 u_x - 9\Delta^3 a_x^{n-5}(a_\sigma^2)^2 (a_\sigma u_\sigma) u_x^2 \\ &+ 15\Delta^2\varphi(\Delta_\sigma u_\sigma) u_x^2 + 2\Delta^3\varepsilon u_x^3 - 8\Delta^3 a_x^{n-4} b_x^{n-2} a_\sigma^2(\Delta ab)(abu) u_x^2 - 3\Delta^2\Gamma_1 u_x^3\Big] u_x^9, \end{aligned}$$

in which the second, third, fourth, sixth and ninth terms must be further reduced. We shall refer to these terms as T_2, T_3, etc., for convenience. We have

$$T_2 = \tfrac{8}{3}\Delta^3 a_x^{n-3} b_x^{n-2}[(\Delta ab) u_x + a_x(\Delta bu) - b_x(\Delta au)](abu)(a_\sigma u_\sigma) = T_{21} + T_{23},$$

the second term containing $(buu) = 0$. For the first term we have

$$\begin{aligned} T_{21} &= \tfrac{8}{3}\Delta^2 a_x^{n-3} b_x^{n-2}(\Delta ab)[(\Delta ab) u_x + a_x(\Delta bu) - b_x(\Delta au)](a_\sigma u_\sigma) u_x \\ &= \tfrac{8}{3}\Delta^2 a_x^{n-3} b_x^{n-2}(\Delta ab) a_\sigma[\Delta_\sigma(abu) - a_\sigma(\Delta bu) + b_\sigma(\Delta au)] u_x^2 \\ &\quad - \tfrac{8}{9}\Delta^3 b_x^{n-2}(\Delta bu)^2 u_x - \tfrac{8}{3}\Delta^2 a_x^{n-3} a_\sigma[\Delta_\sigma a_y - a_\sigma\Delta_y + f_\sigma(\Delta au)](\Delta au) u_x, \end{aligned}$$

where the second, third and sixth are zero terms and the fourth and last terms are alike and equal to

$$-\tfrac{16}{9}\Delta^3 a_x^{n-2}(\Delta au)^2 u_x.$$

The other two terms are equal to

$$\begin{aligned} &\tfrac{8}{3}\Delta a_x^{n-3} b_x^{n-2}(\Delta_\sigma a_\sigma)(\Delta ab)[(\Delta ab) u_x + a_x(\Delta bu) - b_x(\Delta au)] u_x^2 \\ &\quad - \tfrac{8}{3}\Delta a_x^{n-3}(\Delta_\sigma a_\sigma)[a_x\Delta_y + (\Delta af) u_x](\Delta au) u_x, \end{aligned}$$

the second and fourth of which are zero terms. We thus have

$$T_{21} = \tfrac{8}{3}\Delta a_x^{n-3} b_x^{n-2}(\Delta_\sigma a_\sigma)(\Delta ab)^2 u_x^3 - \tfrac{16}{3}\Delta a_x^{n-3}(\Delta_\sigma a_\sigma)(\Delta af)(\Delta au) u_x^2 - \tfrac{16}{9}\Delta^3 a_x^{n-2}(\Delta au)^2 u_x.$$

The other term in T_2 is equal to

$$T_{23} = -\tfrac{8}{3}\Delta^2 a_x^{n-3}[a_x\Delta_y + (\Delta af)u_x](\Delta au)(a_\sigma u_\sigma),$$

the first term of which contains $(\Delta uu) = 0$. Proceeding with the other term we have

$$T_{23} = -\tfrac{8}{3}\Delta^2 a_x^{n-3} a_\sigma[\Delta_\sigma a_y - a_\sigma\Delta_y + f_\sigma(\Delta au)](\Delta au)u_x,$$

the second term of which is zero, the last term equal to

$$-\tfrac{8}{9}\Delta^3 a_x^{n-2}(\Delta au)^2 u_x,$$

and the first term equal to

$$-\tfrac{8}{3}\Delta a_x^{n-3}(\Delta_\sigma a_\sigma)[a_x\Delta_y + (\Delta af)u_x](\Delta au)u_x.$$

This first term is zero. Collecting terms we have

$$T_2 = \tfrac{8}{3}\Delta a_x^{n-3} b_x^{n-2}(\Delta_\sigma a_\sigma)(\Delta ab)^2 u_x^3 - 8\Delta a_x^{n-3}(\Delta_\sigma a_\sigma)(\Delta af)(\Delta au)u_x^2 - \tfrac{8}{3}\Delta^3 a_x^{n-2}(\Delta au)^2 u_x.$$

In the same way T_3, T_4 and T_9 are reduced, and denoting the sum of T_2, T_3, T_4, and T_9 by T, we have

$$\begin{aligned} T = {} & \tfrac{32}{3}\Delta a_x^{n-3} b_x^{n-2}(\Delta_\sigma a_\sigma)(\Delta ab)^2 u_x^3 - 16\Delta^2 a_x^{n-4} b_x^{n-2} a_\sigma^2(\Delta ab)^2 u_x^3 \\ & - 24\Delta a_x^{n-3}(\Delta_\sigma a_\sigma)(\Delta af)(\Delta au)u_x^2 + 24\Delta^2 a_x^{n-4} a_\sigma^2(\Delta af)(\Delta au)u_x^2 \\ & - 8\Delta^3 a_x^{n-2}(\Delta au)^2 u_x. \end{aligned}$$

This last term cancels the fifth term in $A_0^2 C_4$. The first two terms were found in the reduction of G^2 to be equal to

$$\tfrac{8}{3}\Delta^2\left(4\Gamma_2 - 6\Delta\Omega_1 + \tfrac{7}{3}\Delta\psi\right)u_x^3.$$

The other two terms will have to be further reduced. We have from equations (16),

$$\begin{aligned} \eta\Delta_y &= a_x^{n-3} b_x^{n-2}(\Delta ab)^2(\Delta af)\Delta_y = a_x^{n-3} b_x^{n-2}(\Delta ab)(\Delta af)[(\Delta af)(\Delta bu) - (\Delta bf)(\Delta au)] \\ &= -\tfrac{1}{2} a_x^{n-3} b_x^{n-2}(\Delta ab)[\Delta^2 a_\sigma^2 - 2\Delta a_x(\Delta_\sigma a_\sigma) + \theta a_x^2](\Delta bu) \\ &\quad - \tfrac{1}{2} a_x^{n-3} b_x^{n-2} c_x^{n-2}(\Delta af)(\Delta bc)[\Delta(abc) - a_x(\Delta bc)](\Delta au), \end{aligned}$$

in which the first two terms are zero, the third and last terms are alike and their sum is equal to

$$-\tfrac{1}{2}\theta a_x^{n-2} b_x^{n-2}(\Delta ab)[\Delta(abu) - (\Delta ab)u_x] = -\tfrac{1}{2}\Delta\theta(\Delta_\sigma u_\sigma) + \tfrac{1}{2}\theta^2 u_x.$$

We thus have

$$\Delta a_x^{n-3}(\Delta_\sigma a_\sigma)(\Delta af)(\Delta au) = -2\eta\Delta_y - \Delta\theta(\Delta_\sigma u_\sigma) + \theta^2 u_x,$$

from which we get a reduction for the third term in T. Also from (16) we have

$$\begin{aligned} \zeta\Delta_y &= a_x^{n-4} a_\sigma^2(\Delta_\sigma a_\sigma)(\Delta af)\Delta_y = a_x^{n-4} a_\sigma^2(\Delta af)\Delta_\sigma[\Delta_\sigma a_y + f_\sigma(\Delta au) - u_\sigma(\Delta af)] \\ &= -\tfrac{1}{2}\theta a_x^{n-4} a_\sigma^2[a_x^2(\Delta_\sigma u_\sigma) - \Delta a_x(a_\sigma u_\sigma) - a_x(\Delta_\sigma a_\sigma)u_x + \Delta a_\sigma^2 u_x] \\ &\quad + \tfrac{1}{3}\Delta^2 a_x^{n-4} a_\sigma^2(\Delta af)(\Delta au) + \tfrac{1}{2} a_x^{n-4} a_\sigma^2[\Delta^2 a_\sigma^2 - 2\Delta a_x(\Delta_\sigma a_\sigma) + \theta a_x^2](\Delta_\sigma u_\sigma), \end{aligned}$$

in which the first term cancels the last term and the second term cancels half of the

next to the last term. We thus have

$$\Delta^2 a_x^{n-4} a_\sigma^2 (\Delta a f)(\Delta a u) = 3\zeta \Delta_y + \tfrac{3}{2}\Delta\theta(\Delta_\sigma u_\sigma) - \tfrac{3}{2}\theta(\theta - \Delta\varphi)u_x,$$

from which we get a reduction for the fourth term in T.

Making these substitutions, we get

$$T = -8\Delta^3 a_x^{n-2}(\Delta a u)^2 u_x + 24(2\eta + 3\zeta)\Delta_y u_x^2 + 12\Delta(5\theta - 3\Delta\varphi)(\Delta_\sigma u_\sigma)u_x^2 - 12\theta(5\theta - 3\Delta\varphi)u_x^3 + \tfrac{8}{3}\Delta^2\left(4\Gamma_2 - 6\Delta\Omega_1 + \tfrac{7}{3}\Delta\psi\right)u_x^3.$$

The sixth term in $A_0^2 C_4$ is equal to

$$T_6 = -9\Delta^2 a_x^{n-5}(a_\sigma^2)^2[a_x(\Delta_\sigma u_\sigma) + 2b_x^{n-2}(\Delta a b)b_y]u_x^2$$
$$= -9\Delta^2\varphi(\Delta_\sigma u_\sigma)u_x^2 - 18\Delta a_x^{n-5} b_x^{n-2}(a_\sigma^2)^2(\Delta a b)[b_x\Delta_y + (\Delta b f)u_x]u_x^2,$$

the last term of which is equal to

$$-9\Delta a_x^{n-5} b_x^{n-2} c_x^{n-2}(a_\sigma^2)^2(\Delta b c)[\Delta(a b c) - a_x(\Delta b c)]u_x^3$$
$$= -9\Delta^2\Gamma_1 u_x^3 + 9\Delta\varphi\theta u_x^3.$$

We thus have

$$T_6 = -9\Delta^2\varphi(\Delta_\sigma u_\sigma)u_x^2 - 18\Delta\zeta\Delta_y u_x^2 + 9\Delta(\varphi\theta - \Delta\Gamma_1)u_x^3.$$

Now substituting these values for T and T_6, we have

$$(31)\qquad A_0^2 C_4 = \frac{n^4(n-1)^4}{2^{12}\cdot 3^5\cdot 5}[-90\Delta^2 H u_\sigma^2 - 36 G\Delta_y u_x + 180\Delta H(\Delta_\sigma u_\sigma)u_x - 90\theta H u_x^2 - 45 H^2 u_x^2 + \Delta^2 I u_x^2]u_x^{10},$$

where we have put

$$(32)\qquad \begin{cases} I = \Delta I_0 + I_1 \\ I_0 = 12\varepsilon - 96\Omega_1 + \frac{112}{3}\psi \\ I_1 = 45\varphi^2 - 72\Gamma_1 + 64\Gamma_2. \end{cases}$$

For the second term in B_4, by equations (18) and (24), we have

$$-3A_1 B_3 = \frac{n^4(n-1)^4}{2^{10}\cdot 3^3\cdot 5} G\Delta_y u_x^{11},$$

and for the third term, by equations (18) and (21), we have

$$-3A_1^2 B_2 = \frac{n^4(n-1)^4}{2^{11}\cdot 3^3}[\Delta^2 u_\sigma^2 - 2\Delta(\Delta_\sigma u_\sigma)u_x + \theta u_x^2]H u_x^{10}.$$

Adding the results for the three terms in B_4, we get

$$(33)\qquad B_4 = \frac{n^4(n-1)^4}{2^{12}\cdot 3^5\cdot 5}(\Delta^2 I - 45 H^2)u_x^{12}.$$

Making $n = 6$ in equation (5) and equating the discriminant to zero, we have

$$(B_4 + 3B_2^2)^3 - 27(B_2 B_4 - B_3^2 - B_2^3)^2 = 0.$$

Substituting in this equation the expressions given for B_2, B_3 and B_4 in equations (21), (24) and (33), we get

$$(34)\qquad 5\Delta^6 I^3 - 81[5\Delta^2 H I - 12(G^2 + 25H^3)]^2 = 0,$$

for the equation of a bitangential curve of the general plane sextic curve, but containing the sixth power of the hessian of the sextic as an extraneous factor.

§ 10.

Removal of the extraneous factor Δ^6.

The removal of the extraneous factor Δ^6 is easily accomplished. Referring to the expression for $G^2 + 25H^3$, given in equation (28), we see that we already have Δ^4 removed and that the removal of the remaining Δ^2 requires that the terms in $5HI$ which do not contain Δ must cancel the terms within the braces in equation (28) which do not contain Δ. The terms in $5HI$ which do not contain Δ explicitly are

$$450\varphi^2\theta - 720\theta\Gamma_1 + 640\theta\Gamma_2,$$

and the terms in $-12(G^2 + 25H^3)$ which do not contain Δ explicitly are

$$-450\varphi^2\theta + 720\theta\Gamma_1 - 1920\theta\Gamma_2,$$

from which it appears that $\theta\Gamma_2$ must contain Δ as a factor. To prove this we have

$$\begin{aligned}
\theta\Gamma_2 &= \alpha_x^{3n-8} a_x^{n-3} b_x^{n-2} (\Delta_\sigma a_\sigma)(\Delta_\sigma \Delta_\sigma)(\alpha a b)^2 \\
&= \alpha_x^{3n-8} a_x^{n-3} b_x^{n-2} (\Delta_\sigma a_\sigma)(\alpha a b)\Delta_\sigma[\alpha_\sigma(\Delta a b) - a_\sigma(\Delta \alpha b) + b_\sigma(\Delta \alpha a)] \\
&= \alpha_x^{3n-8}\beta_x^{3n-8} a_x^{n-3} b_x^{n-2} (\Delta_\sigma a_\sigma)(\alpha_\sigma\beta_\sigma)(\alpha a b)[\Delta(\beta a b) + a_x(\Delta\beta b) - b_x(\Delta\beta a)] \\
&\quad + \theta\alpha_x^{3n-8} a_x^{n-2}(\Delta\alpha a)^2 + \tfrac{2}{3}\Delta\alpha_x^{3n-8} a_x^{n-3} b_x^{n-2} c_x^{n-2}(\alpha a b)(\Delta\alpha b)(\Delta a c)^2 \\
&\quad + \tfrac{1}{3}\Delta\alpha_x^{3n-8} a_x^{n-3}(\Delta_\sigma a_\sigma)(\Delta\alpha a)^2.
\end{aligned}$$

Here, the first two and last three terms are easily reduced to

$$\tfrac{1}{3}\Delta\Omega_4 + \Delta\varphi'\Gamma_2 - \tfrac{1}{6}\Delta\theta\psi - \tfrac{1}{6}\Delta^2\varphi'\psi,$$

and the third term can be written as

$$T = \alpha_x^{3n-8}\beta_x^{3n-8}\gamma_x^{3n-8} a_x^{n-3}(\Delta_\sigma a_\sigma)(\alpha_\sigma\beta_\sigma)(\beta\gamma a)[\alpha_x(\gamma a f) + a_x(\alpha\gamma f)] = T_1 + T_2.$$

For these two terms we have

$$\begin{aligned}
T_1 &= \tfrac{1}{2}\beta_x^{3n-8}\gamma_x^{3n-8} a_x^{n-3}(\Delta_\sigma a_\sigma)(\beta\gamma a)\Delta_\sigma[f_\sigma(\beta\gamma a) - a_\sigma(\beta\gamma f)] \\
&= \tfrac{1}{6}\Delta^2 a_x^{n-3} a_\omega^2(\Delta_\sigma a_\sigma) - \tfrac{1}{2}\theta a_x^{n-3} a_\sigma^2(a_\omega f_\omega) + \tfrac{1}{3}\Delta a_x^{n-3} b_x^{n-2}(a_\omega f_\omega)(\Delta a b)^2 \\
&= \tfrac{1}{6}\Delta^2\Lambda + \tfrac{1}{3}\Delta\Omega_4,
\end{aligned}$$

the middle term being equal to zero, and

$$\begin{aligned}
T_2 &= \tfrac{1}{3}\Delta\alpha_x^{3n-8}\beta_x^{3n-8}\gamma_x^{3n-8}(\alpha_\sigma\beta_\sigma)(\beta\gamma\Delta)(\alpha\gamma f) \\
&= \tfrac{1}{6}\Delta\alpha_x^{3n-8}\beta_x^{3n-8}\gamma_x^{3n-8}(\beta\gamma\Delta)\alpha_\sigma[\alpha_\sigma(\beta\gamma f) - f_\sigma(\alpha\beta\gamma)],
\end{aligned}$$

the first term of which is zero and the last term of which is equal to

$$-\tfrac{1}{18}\Delta^2\Delta_\omega^2 = -\tfrac{1}{54}\Delta^3\Delta_\Delta.$$

Collecting terms, we have

$$\theta\Gamma_2 = \tfrac{1}{2}\Delta\chi,$$

where we have put

$$\chi = \tfrac{4}{3}\Omega_4 + \tfrac{1}{3}\Lambda + 2\varphi'\Gamma_2 - \tfrac{1}{3}\theta\psi - \tfrac{1}{3}\Delta\varphi'\psi - \tfrac{1}{27}\Delta^2\Delta_\Delta. \tag{35}$$

Using this expression for $\theta\Gamma_2$ and substituting in equation (34) the values of HI and $G^2 + 25H^3$ given in equations (21), (28) and (32), we get

$$\Delta^6\left\{5I^3 - 81\left[\begin{matrix}5(HI_0 - \varphi I_1 - 128\chi)\\ -12(15\varphi\Gamma - 25\varphi^3 + 8\Omega + \Delta\Upsilon)\end{matrix}\right]^2\right\} = 0 \tag{36}$$

for the equation of a bitangential curve of the general plane sextic curve with the extraneous factor Δ^6 removed.

§ 11.

The equation of a bitangential of the septimic.

The processes involved in finding the equation of a bitangential curve of the general plane septimic curve are the same, with very few exceptions, as they are in the case of the sextic. We shall therefore give only the results obtained in the several steps. No attempt will be made, in this paper, to remove the extraneous factor Δ^{12} from the equation here obtained.

From equation (7) we have

$$B_5 = A_0^3 C_5 - 4A_1B_4 - 6A_1^2B_3 - 4A_1^3B_2,$$

from which to compute a form for B_5. Making $t=3$ in equation (20) and performing the multiplications indicated, introducing the value of A_0^3 from equation (17) and carrying the several steps through in a manner similar to the method used in the case of the sextic, we finally get for the first term in B_5 the equation

$$\left\{\begin{aligned} A_0^3 C_5 = \frac{n^5(n-1)^5}{2^{12}\cdot 3^6\cdot 5\cdot 7}\big[&210\Delta^2 H\Delta_y u_\sigma^2 - 63\Delta^2 G u_\sigma^2 u_x - 420\Delta H(\Delta_\sigma u_\sigma)\Delta_y u_x\\ &+ 126\Delta G(\Delta_\sigma u_\sigma)u_x^2 - (7\Delta^2 I - 315H^2)\Delta_y u_x^2 + 210H\theta\Delta_y u_x^2\\ &- 84G\theta u_x^3 + \Delta J u_x^3\big]u_x^{12},\end{aligned}\right. \tag{37}$$

where we have put

$$\left\{\begin{aligned} J &= \Delta^2 J_0 + \Delta J_1 + J_2,\\ J_0 &= -15\Pi_1, \qquad J_1 = 120\Pi_2 + 90\Pi_3,\\ J_2 &= -162\xi\theta - 48\Pi_4\theta + 48\Pi_5\theta - 80\Pi_6.\end{aligned}\right. \tag{38}$$

For the other three terms in B_5 we have

$$-4A_1B_4 = \frac{n^5(n-1)^5}{2^{12}\cdot 3^6\cdot 5\cdot 7}[7\Delta^2 I - 315H^2]\Delta_y u_x^{14},$$

$$-6A_1^2B_3 = \frac{n^5(n-1)^5}{2^{12}\cdot 3^6\cdot 5\cdot 7}[63\Delta^2 u_\sigma^2 - 126\Delta(\Delta_\sigma u_\sigma)u_x + 63\theta u_x^2]Gu_x^{13},$$

$$-4A_1^3B_2 = \frac{n^5(n-1)^5}{2^{12}\cdot 3^6\cdot 5\cdot 7}[-210\Delta^2 u_\sigma^2 + 420\Delta(\Delta_\sigma u_\sigma)u_x - 210\theta u_x^2]H\Delta_y u_x^{12}.$$

Adding these three terms to the first term as given in equation (37), we get

$$B_5 = \frac{n^5(n-1)^5}{2^{12}\cdot 3^6\cdot 5\cdot 7}[\Delta J - 21\,G\theta]u_x^{15}. \tag{39}$$

Now making $n=7$ in equation (5) and equating the discriminant to zero, we have

$$\begin{aligned} &B_5^4 - 120\,B_5^3 B_3 B_2 + 8\,B_5^2[20\,B_3^2 B_2 + 45\,B_4(B_3^2 - 4\,B_2^3) + 6\,B_2^2(55\,B_3^2 + 72\,B_2^3)] \\ &+ 32\,B_5 B_3[-20\,B_4^3 + 140\,B_4^2 B_2^2 - 45\,B_4 B_2(7\,B_3^2 + 8\,B_2^3) + 4\,B_2^2(27\,B_3^2 + 40\,B_2^3)] \\ &+ 16\,B_4^2[16\,B_4^3 - 160\,B_4^2 B_2^2 + 40\,B_4 B_2(9\,B_3^2 + 10\,B_2^3) - 5\,B_2^3(27\,B_3^2 + 40\,B_2^3)] = 0. \end{aligned}$$

Substituting in this equation the forms for B_2, B_3, B_4 and B_5 given in equations (21), (24), (33) and (39), we get

$$\left\{\begin{aligned} &97200(\Delta J - 21\,G\theta)^4 - 20412000(\Delta J - 21\,G\theta)^3 GH \\ &+ 264600(\Delta J - 21\,G\theta)^2[5(\Delta^2 I - 45\,H^2)^2 H + 27(\Delta^2 I - 45\,H^2)(G^2 - 25\,H^3) \\ &+ 270(11\,G^2 H^2 + 90\,H^5)] + 24696(\Delta GJ - 21\,G^2\theta)[-25(\Delta^2 I - 45\,H^2)^3 \\ &+ 2625(\Delta^2 HI - 45\,H^3)^2 - 2025(\Delta^2 HI - 45\,H^3)(7\,G^2 + 50\,H^3) + \\ &+ 432(27\,G^4 + 250\,G^2 H^3)] + 12005(\Delta^2 I - 45\,H^2)^2[(\Delta^2 I - 45\,H^2)^3 \\ &- 150(\Delta^2 HI - 45\,H^3)^2 - 45(\Delta^2 HI - 45\,H^3)(18\,G^2 + 125\,H^3) \\ &- 27(27\,G^4 + 250\,G^2 H^3)] = 0 \end{aligned}\right. \tag{40}$$

for the equation of a bitangential curve of the general plane septimic curve, but containing Δ^{12} as an extraneous factor. Evidently the labor in the removal of this factor would be tedious.

§ 12.

The equation of a bitangential of the octic.

The processes in the case of the octic are very little different from those of the sextic and septimic. We have from equation (7),

$$B_6 = A_0^4 C_6 - 5\,A_1 B_5 - 10\,A_1^2 B_4 - 10\,A_1^3 B_3 - 5\,A_1^4 B_2,$$

from which to find a form for B_6. Making $t=3$ in equation (19) and performing the multiplications indicated, introducing the value of A_0^4 from equation (17) and carrying the several steps through as we did in the case of the sextic, we get for the first term in B_6 the equation

$$\left\{\begin{aligned} A_0^4 C_6 = \frac{n^6(n-1)^6}{2^{17}\cdot 3^7\cdot 7}&[210\,\Delta^4 H(u_\sigma^2)^2 + 168\,\Delta^2 G\Delta_y u_\sigma^2 u_x - 840\,\Delta^3 H(\Delta_\sigma u_\sigma)u_\sigma^2 u_x \\ &- 14\,\Delta^2(45\,H^2 + 90\,H\theta - \Delta^2 I)u_\sigma^2 u_x^2 - 560\,\Delta^2 H a_x^{n-2}(\Delta a u)^2 u_x^2 \\ &- 336\,\Delta G(\Delta_\sigma u_\sigma)\Delta_y u_x^2 + 8(42\,G\theta - \Delta J)\Delta_y u_x^3 \\ &+ 28\,\Delta(\Delta^2 I - 30\,H\theta - 45\,H^2)(\Delta_\sigma u_\sigma)u_x^3 + (1632\,H\theta^2 - \Delta K)u_x^4]u_x^{14}, \end{aligned}\right. \tag{41}$$

where we have put

$$(42)\qquad \begin{cases} K = \Delta^4 K_0 + \Delta^3 K_1 + \Delta^2 K_2 + \Delta K_3 + K_4, \\ K_0 = -9\Sigma_1, \qquad K_1 = 72\Sigma_2 + 144\Sigma_3, \\ K_2 = 672\Omega_1\theta - \frac{464}{3}\psi\theta - 420\varepsilon\theta - 64\Sigma_4 - 192\Sigma_5, \\ K_3 = 1512\Gamma_1\theta - 992\Gamma_2\theta, \qquad K_4 = -816\varphi\theta. \end{cases}$$

For the other four terms in B_6 we have

$$-5A_1B_5 = \frac{n^6(n-1)^6}{2^{17}.3^7.7}[8\Delta J - 168G\theta]\Delta_y u_x^{17},$$

$$-10A_1^2B_4 = \frac{n^6(n-1)^6}{2^{17}.3^7.7}[\Delta^2 u_\sigma^2 - 2\Delta(\Delta_\sigma u_\sigma)u_x + \theta u_x^2][14\Delta^2 I - 630H^2]u_x^{16},$$

$$-10A_1^3B_3 = \frac{n^6(n-1)^6}{2^{17}.3^7.7}[-\Delta^2 u_\sigma^2 + 2\Delta(\Delta_\sigma u_\sigma)u_x - \theta u_x^2]168G\Delta_y u_x^{15},$$

$$-5A_1^4B_2 = \frac{n^6(n-1)^6}{2^{17}.3^7.7}[-3\Delta^4(u_\sigma^2)^2 + 12\Delta^3(\Delta_\sigma u_\sigma)u_\sigma^2 u_x - 18\Delta^2\theta u_\sigma^2 u_x^2 + 8\Delta^2 a_x^{n-2}(\Delta a u)^2 u_x^2 + 12\Delta\theta(\Delta_\sigma u_\sigma)u_x^3 - 3\theta^2 u_x^4]70Hu_x^{14}.$$

Adding these four terms to the first term as given in equation (41), we get

$$(43)\qquad B_6 = \frac{n^6(n-1)^6}{2^{17}.3^7.7}\left\{\begin{matrix} 14\Delta^2\theta I - \Delta K \\ -630H^2\theta + 1422H\theta^2 \end{matrix}\right\}u_x^{18}.$$

Now making $n=8$ in equation (5) and equating the discriminant to zero, we get an equation consisting of seventy-seven terms made up of the coefficients B_2, B_3, B_4, B_5 and B_6, in which we can substitute the forms given by equations (21), (24), (33), (39) and (43) and thus get the equation of a bitangential curve of the general plane octic curve, but containing the twentieth power of the hessian of the octic as an extraneous factor.

Bloomington, Ind. (U.S.A.), January 16th, 1909.

U. S. Hanna.